The maximum spectral radius of C_4-free graphs of given order and size

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Abstract

Suppose that G is a graph with n vertices and m edges, and let μ be the spectral radius of its adjacency matrix.

Recently we showed that if G has no 4-cycle, then $\mu^2 - \mu \leq n-1$, with equality if and only if G is the friendship graph.

Here we prove that if $m \geq 9$ and G has no 4-cycle, then $\mu^2 \leq m$, with equality if G is a star. For $4 \leq m \leq 8$ this assertion fails.

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This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [3] and [6, 14].

Suppose G is a graph with n vertices and m edges and let $\mu(G)$ be the spectral radius of its adjacency matrix. How large can $\mu(G)$ be if G has no cycles of length 4? This question was partially answered in [10], Theorem 3:

Let G be a graph of order n with $\mu(G) = \mu$. If G has no 4-cycles, then

$$\mu^2 - \mu \leq n-1. \tag{1}$$

Equality holds if and only if every two vertices of G have exactly one common neighbor.

The condition for equality in (1) is a popular topic: as shown in [4] and [5], the only graph satisfying this condition is the *friendship graph* - a set of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex. Thus equality is possible only for n odd, and (1) may be improved for even n.

Conjecture 1 *Let G be a graph of even order n with $\mu(G) = \mu$. If G has no 4-cycles, then*

$$\mu^3 - \mu^2 - (n-1)\mu + 1 \leq 0. \tag{2}$$

Equality holds if and only if G is a star of order n with $n/2 - 1$ disjoint additional edges.

Note that the number of edges of G is missing in (1) and (2). In contrast, Nosal [15] showed that if $\mu(G) > \sqrt{m}$, then G has triangles. Our main result here is a similar assertion for 4-cycles:

Theorem 2 *Let $m \geq 9$ and G be a graph with m edges. If $\mu(G) > \sqrt{m}$, then G has a 4-cycle.*

Note that Theorem 2 is tight, for all stars are C_4-free graphs with $\mu(G) = \sqrt{m}$. Also, let $S_{n,1}$ be the star of order n with an edge within its independent set: $S_{n,1}$ is C_4-free and has n edges, but $\mu(G) > \sqrt{n}$ for $4 \leq n \leq 8$, as shown in Lemma 6 below.

Observe that the original result of Nosal was sharpened in [12], Theorem 2, (i):

If $\mu(G) \geq \sqrt{m}$, then G has a triangle, unless G is a complete bipartite graph with possibly some isolated vertices.

It turns out that Theorem 2 can be sharpened likewise, at the price of a considerably longer proof, which we omit.

Theorem 3 *Let $m \geq 9$ and G be a graph with m edges. If $\mu(G) \geq \sqrt{m}$, then G has a 4-cycle unless G is a star or $S_{9,1}$ with possibly some isolated vertices.*

Proofs

Our notation follows [2]; thus, if G is a graph G, and X and Y are disjoint sets of vertices of G, we write:

- $E(G)$ for the edge set of G and $e(G)$ for $|E(G)|$;
- $G[X]$ for the graph induced by X, $E(X)$ for $E(G[X])$, and $e(X)$ for $|E(X)|$;
- $e(X,Y)$ for the number of edges joining vertices in X to vertices in Y;
- $G - uv$ for the graph obtained by removing the edge $uv \in E(G)$;
- $\Gamma_G(u)$ for the set of neighbors of a vertex u and $d_G(u)$ for $|\Gamma_G(u)|$;
- $\Gamma_X(u)$ for $\Gamma_G(u) \cap X$ and $d_X(u)$ for $|\Gamma_X(u)|$.

We drop the subscript in $\Gamma_G(u)$ and $d_G(u)$ when it is understood.

Define $S_{n,k}$ to be the star of order n with k disjoint edges within its independent set.

Next we give some facts, needed in the proof of Theorem 2.

First, a fact implied by Theorem 1 in [16]:

Fact 4 *Let $\mathbf{x}$ be a unit eigenvector to the spectral radius of a graph with some edges. Then the entries of $\mathbf{x}$ do not exceed $2^{-1/2}$.* $\square$

Next, a known fact, proved here for completeness:

Lemma 5 *Let A and A' be the adjacency matrices of two graphs G and G' on the same vertex set. Suppose that $\Gamma_G(u) \subsetneq \Gamma_{G'}(u)$ for some vertex u. If some positive eigenvector $\mathbf{x}$ to $\mu(G)$ satisfies $\langle A'\mathbf{x}, \mathbf{x}\rangle \geq \langle A\mathbf{x}, \mathbf{x}\rangle$, then $\mu(G') > \mu(G)$.*

Proof Since $\langle A'\mathbf{x},\mathbf{x}\rangle \geq \langle A\mathbf{x},\mathbf{x}\rangle$, the Rayleigh principle implies that $\mu(G') \geq \mu(G)$. If $\mu(G') = \mu(G)$, then $\langle A'\mathbf{x},\mathbf{x}\rangle = \langle A\mathbf{x},\mathbf{x}\rangle$, and, again by the Rayleigh principle, $\mathbf{x}$ is an eigenvector to $\mu(G')$. But this is impossible, for

$$\mu(G')\, x_u = \sum_{uv \in E(G')} x_v > \sum_{uv \in E(G)} x_v = \mu(G)\, x_u.$$

We use above that $\Gamma_G(u) \subset \Gamma_{G'}(u)$, but there is some $v \in \Gamma_{G'}(u)$ such that $v \notin \Gamma_G(u)$. This completes the proof of Lemma 5. $\square$

Finally, some facts about $\mu(S_{n,k})$:

Lemma 6 *(a)* $\mu(S_{n,k})$ *is the largest root of the equation*

$$x^3 - x^2 - (n-1)x + n - 1 - 2k = 0;$$

(b) $\mu(S_{n,k}) \leq \sqrt{n-1+k}$ *for* $n-1+k \geq 9$, *and* $\mu(S_{n,1}) > \sqrt{n}$ *for* $4 \leq n \leq 8$.

Proof Suppose that 1 is the dominating vertex of $S_{n,k}$, and $\{2,3\},\ldots,\{2k,2k+1\}$ are its k additional edges. Set $\mu = \mu(S_{n,k})$ and let $(x_1,\ldots,x_n)$ be an eigenvector to μ. By symmetry,

$$x_2 = x_3 = \cdots = x_{2k+1} \quad \text{and} \quad x_{2k+2} = x_{2k+3} = \cdots = x_n.$$

Setting $x_1 = x$, $x_2 = y$, $x_n = z$, we see that

$$\begin{aligned} \mu z &= x, \\ \mu y &= y + x, \\ \mu x &= 2ky + (n-2k-1)\, z. \end{aligned}$$

Solving this system, we find that μ is a root of the equation

$$x^3 - x^2 - (n-1)x + n - 1 - 2k = 0.$$

If μ is not the largest root of this equation, then it has to be smaller than

$$x_{\min} = 1/3 + \sqrt{1/9 + (n-1)/3},$$

the point where the function

$$f_k(x) = x^3 - x^2 - (n-1)x + n - 1 - 2k$$

has a local minimum. This, however, is not possible since

$$\mu > \sqrt{n-1} > 1/3 + \sqrt{1/9 + (n-1)/3}.$$

This completes the proof of *(a)*,

To prove *(b)* note that

$$\begin{aligned} f_k\left(\sqrt{n-1+k}\right) &= \left(\sqrt{n-1+k}\right)^3 - \left(\sqrt{n-1+k}\right)^2 - (n-1)\sqrt{n-1+k} + n - 2k \\ &= k\left(\sqrt{n-1+k} - 3\right), \end{aligned}$$

implying the assertion since $\sqrt{n-1+k} > x_{\min}$ and $f_k(x)$ is increasing for $x > x_{\min}$. $\square$

Proof of Theorem 2

Let $m \geq 9$, and assume for a contradiction that G is a C_4-free graph with m edges, satisfying $\mu(G) > \sqrt{m}$. Set $\mu = \mu(G)$, and suppose that

$$\mu = \max\{\mu(G) : G \text{ is a } C_4\text{-free graph with } e(G) = m\}. \tag{3}$$

Also, for the purposes of the proof we may and shall suppose that G has no isolated vertices. This implies that G is connected.

Indeed, let G_1 be a component of G with $\mu(G_1) = \mu(G)$, and let G_2 be the nonempty union of the remaining components of G. Remove an edge from G_2, and add an edge between G_1 and G_2. The resulting graph is C_4-free with m edges, but its spectral radius is larger than μ, contradicting (3). Hence, G is connected.

The essentially part of the proof is induction on m, but it needs some preparation. We first introduce some structure in G and settle several cases with direct arguments, in particular the case $m \leq 13$. Then, having restricted the structure of G, we prove the induction step. Now the details.

Let $\{1, \ldots, n\}$ be the vertices of G, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a positive unit eigenvector to μ, i.e.,

$$\mu = 2 \sum_{ij \in E(G)} x_i x_j.$$

By symmetry, suppose that $x_1 \geq \cdots \geq x_n$. We claim that all vertices of degree 1 are joined to vertex 1.

Indeed, assume for a contradiction that there exists a vertex $u \neq 1$ such that $d(u) = 1$ and u is joined to $v \neq 1$. Remove the edge uv and join u to vertex 1. The resulting graph G' is C_4-free and has m edges. Also, we see that

$$\sum_{ij \in E(G')} x_i x_j = \sum_{ij \in E(G)} x_i x_j + x_u(x_1 - x_v) \geq \sum_{ij \in E(G)} x_i x_j.$$

Since $\Gamma_G(1) \subsetneqq \Gamma_{G'}(1)$, Lemma 5 implies that $\mu(G') > \mu$, contradicting (3). Hence, all vertices of degree 1 are joined to vertex 1.

Let $A = (a_{ij})$ be the adjacency matrix of G and $A^2 = B = (b_{ij})$. Since $\mathbf{x}$ is an eigenvector of B to μ^2, we have

$$x_1 \mu^2 = \sum_{i=1}^{n} b_{1i} x_i \leq x_1 \sum_{i=1}^{n} b_{1i} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{1j} a_{ji} = x_1 \sum_{v \in \Gamma(1)} d(v). \tag{4}$$

Set

$$U = \Gamma(1), \qquad W = \{2, 3, \ldots, n\} \backslash \Gamma(1),$$

and let $t = e(U)$ and $q = e(W)$. We see that

$$\sum_{v \in U} d(v) = d(1) + 2e(U) + e(U, W) = e(G) - e(W) + e(U) = m - q + t.$$

Thus (4) gives $\mu^2 \leq m + t - q$, and from $\mu^2 > m$, we get the crucial inequality $t \geq q + 1$.

Since all vertices of degree 1 belong to U, we have $d(u) \geq 2$ for all $u \in W$. Also, since G is C_4-free, a vertex in W can be joined to at most one vertex in U. Thus, for all $w \in W$ we have $d_W(w) \geq d(w) - 1 \geq 1$, and consequently,

$$2q = \sum_{w \in W} d_W(w) \geq \sum_{w \in W} 1 = |W|.$$

Suppose first that $q = 0$. Then $|W| = 0$, and so, $e(U, W) = 0$. Therefore, vertex 1 is dominating and $G = S_{m+1-t,t}$. By Lemma 6,

$$\mu = \mu(S_{m+1-t,t}) \leq \sqrt{m}$$

for $m \geq 9$, contradicting the hypothesis. Therefore, $q \geq 1$.

The next claim gives a useful property of $G[W]$, and, in particular, settles the case $q = 1$.

Claim 1 *The graph $G[W]$ contains no isolated edges.*

Proof Let $uv \in E(W)$ be an isolated edge. Since $d(u) \geq 2$ and $d(v) \geq 2$, we see that $d_U(u) = d_U(v) = 1$. Let $\{k\} = \Gamma_U(u)$ and $\{l\} = \Gamma_U(v)$. Remove the edges uk, vl, and join u and v to the vertex 1. The resulting graph G' is C_4-free and has m edges. Also, we see that

$$\sum_{ij \in E(G')} x_i x_j = \sum_{ij \in E(G)} x_i x_j + x_u(x_1 - x_k) + x_v(x_1 - x_l) \geq \sum_{ij \in E(G)} x_i x_j.$$

Since $\Gamma_G(1) \subsetneqq \Gamma_{G'}(1)$, Lemma 5 implies that $\mu(G') > \mu$, contradicting (3), and completing the proof of Claim 1. $\square$

Claim 1 implies that $q \geq 2$. Our next goal is to obtain a contradiction for $m \leq 13$. Indeed, suppose that $m \leq 13$; then $q \geq 2$ gives

$$13 \geq m = 3t + e(U, W) + q \geq 4q + 3 + e(U, W) \geq 11 + e(U, W),$$

which is possible only if $q = 2$, $e(U, W) \leq 2$, and $t = 3$.

The graph $G[W]$ has 2 non-isolated edges, and thus is a path of order 3. Let u, v, w be the vertices of this path and suppose that $uv \in E(W)$ and $vw \in E(W)$. Since $d(u) \geq 2$ and $d(w) \geq 2$, we find that $d_U(u) = d_U(w) = 1$. This, in view of $e(U, W) \leq 2$, gives $e(U, W) = 2$, and so, v has no neighbors in U.

Let $\{k\} = \Gamma_U(u)$ and $\{l\} = \Gamma_U(w)$. Remove the edges uk, wl, uv, and join u, v, w to the vertex 1. The resulting graph G' is C_4-free and has m edges. Also, we see that

$$\sum_{ij \in E(G')} x_i x_j = \sum_{ij \in E(G)} x_i x_j + x_u(x_1 - x_k) + x_w(x_1 - x_l) + x_v(x_1 - x_u) \geq \sum_{ij \in E(G)} x_i x_j.$$

Since $\Gamma_G(1) \nsubseteq \Gamma_{G'}(1)$, Lemma 5 implies that $\mu(G') > \mu$, contradicting (3).

At this point we have proved the theorem for $9 \leq m \leq 13$. Assume now that $m \geq 14$ and that the theorem holds for $m - 1$; we shall prove it for m. The induction step is based on three claims.

Claim 2 *If an edge $uv \in E(G)$ satisfies $d(u) = d(v) = 2$, then $x_u x_v < 1/4\mu$.*

Proof Let $\{i, u\} = \Gamma(v)$ and $\{j, v\} = \Gamma(u)$. From

$$\mu x_u = x_i + x_v \leq x_1 + x_v \quad \text{and} \quad \mu x_v \leq x_1 + x_u \leq x_1 + x_v$$

we see that $x_u + x_v = 2x_1/(\mu - 1)$. Hence, using the AM-QM inequality and Fact 4, we obtain

$$x_u x_v \leq \left(\frac{x_u + x_v}{2}\right)^2 = \frac{x_1^2}{(\mu-1)^2} \leq \frac{1}{2(\mu-1)^2} \leq \frac{1}{4\mu}$$

whenever $\mu^2 \geq 14$. This completes the proof of Claim 2. $\square$

Claim 3 *Let $m \geq 20$. Let the vertices u, v, w satisfy $d(u) = d(w) = 2$ and $d(v) = 3$, and let v be joined to u and w. Then either $x_u x_v < 1/4\mu$ or $x_w x_v < 1/4\mu$.*

Proof We first note that if $x \geq \sqrt{20}$, then

$$\frac{(x^2-2)^2}{x(x+1)(x+2)} > \frac{x^4 - 4x^2}{x(x+1)(x+2)} = \frac{x(x-2)}{x+1} = \frac{x^2 - 4x - 2}{x+1} + 2 > 2. \tag{5}$$

Next, letting $\Gamma(u) = \{i, v\}$, $\Gamma(w) = \{j, v\}$, and $\Gamma(v) = \{k, u, w\}$, we see that

$$\begin{aligned}
\mu x_u &= x_i + x_v \leq x_1 + x_v, \\
\mu x_w &= x_j + x_v \leq x_1 + x_v, \\
\mu x_v &= x_k + x_u + x_w \leq x_1 + x_u + x_w,
\end{aligned}$$

and therefore,

$$\begin{aligned}
\mu(x_u + x_w) &\leq x_1 + 2x_v, \\
\mu x_v &\leq x_1 + x_u + x_w.
\end{aligned}$$

The solution of this system is

$$x_u + x_w \leq 2\frac{\mu+1}{\mu^2-2}x_1, \quad x_v \leq \frac{\mu+2}{\mu^2-2}x_1.$$

Now, assuming $x_u \geq x_w$, and using Fact 4, we obtain

$$x_u x_v \leq \frac{(\mu+1)(\mu+2)}{(\mu^2-2)^2}x_1^2 \leq \frac{(\mu+1)(\mu+2)}{2(\mu^2-2)^2}.$$

Finally, inequality (5) implies that

$$x_u x_v \leq \frac{(\mu+1)(\mu+2)}{2(\mu^2-2)^2} \leq \frac{1}{4\mu}$$

whenever $\mu^2 \geq 20$. This completes the proof of Claim 3. $\square$

Claim 4 *If there exists* $uv \in E(G)$ *satisfying* $x_u x_v \leq 1/4\mu$, *then* $\mu^2(G - uv) > \mu^2 - 1$.

Proof For every edge $uv \in E(G)$, by the Rayleigh principle, we have

$$\mu^2(G - uv) \geq \left(2 \sum_{ij \in E(G-uv)} x_i x_j\right)^2 = (\mu - 2x_u x_v)^2 > \mu^2 - 4\mu x_u x_v \geq \mu^2 - 1,$$

completing the proof of Claim 4. $\square$

Having proved the claims, we proceed with the induction step. If there exists $uv \in E(U)$ with $d(u) = d(v) = 2$, then by Claims 2 and 4 we obtain $\mu(G - uv) > \sqrt{m-1}$; by the induction hypothesis G contains a C_4, .a contradiction.

Hereafter, we assume that $d(u) + d(v) \geq 5$ for all $uv \in E(U)$. For every edge $uv \in E(U)$, let $W_{uv} = \Gamma_W(u) \cup \Gamma_W(v)$. Since a vertex in W can be joined to at most one vertex in U, the sets W_{uv}, $uv \in E(U)$ are disjoint. From

$$2q = 2e(W) = \sum_{w \in W} d_W(w) \geq \sum_{uv \in E(U)} \sum_{w \in W_{uv}} d_W(w) \geq t \min_{uv \in E(U)} \sum_{w \in W_{uv}} d_W(w)$$

we see that there is an edge $uv \in E(U)$ such that $\sum_{w \in W_{uv}} d_W(w) \leq 1$. Then from

$$|W_{uv}| = d(u) + d(v) - 4 \geq 1$$

we conclude that W_{uv} contains a single vertex w, and that $d_W(w) = 1$.

Assume, by symmetry, that w is joined to v. Then, $d(u) = 2$, $d(w) = 2$, and $d(v) = 3$. Now, if $m \geq 20$, Claims 3 and 4 imply either $\mu(G - vw) > \sqrt{m-1}$ or $\mu(G - uv) > \sqrt{m-1}$; by the induction hypothesis G contains a C_4, contradiction.

To complete the proof we have to settle the case when $15 \leq m \leq 19$ and $d(u) + d(v) \geq 5$ holds for all $uv \in E(U)$. We shall show that these conditions also lead to a contradiction.

From

$$e(U, W) = \sum_{uv \in E(U)} d_W(u) + d_W(v) \geq \sum_{uv \in E(U)} (5 - 4) = t$$

and

$$19 \geq m = 3t + e(U, W) + q \geq 3t + t + q \geq 5q + 4 \tag{6}$$

we see that $q \leq 3$ and $t \leq 4$.

Consider first the case $q = 3$. From (6) we find that this is possible only if $m = 19$, $t = 4$, $e(U, W) = 4$. This implies also that $|W| \geq e(U, W) \geq 4$.

$G[W]$ has no isolated vertices and, by Claim 1, it has no isolated edges either. Thus, from $e(W) = 3$ we see that $G[W]$ is a tree of order 4. Now the structure of G is determined: G consists of 4 triangles sharing vertex 1, a tree T of order 4, and a 4-matching joining every vertex of T to a separate triangle.

Select $u \in W$ to be with $d_W(u) = 1$ and let $\{v\} = \Gamma_W(u)$, $\{k\} = \Gamma_U(u)$, $\{l\} = \Gamma_U(v)$. Suppose that $x_k \geq x_l$, remove the edge vl, and add the edge vk. The resulting graph G' is C_4-free and has m edges. Also, we see that

$$\sum_{ij \in E(G')} x_i x_j = \sum_{ij \in E(G)} x_i x_j + x_v (x_k - x_l) \geq \sum_{ij \in E(G)} x_i x_j.$$

Since $\Gamma_G(k) \subsetneqq \Gamma_{G'}(k)$, Lemma 5 implies that $\mu(G') > \mu$, contradicting (3).

The same argument applies when $x_k < x_l$, completing the proof in this case.

Let now $q = 2$. If $t = 4$, then $|W| \geq e(U, W) \geq t = 4$, and so W contains isolated edges, contradicting Claim 1. Hence, $t = 3$, $|W| = 3$, and $G[W]$ is a path of order 3. Now, the structure of G is determined: G consists of the graph $S_{m-4,3}$, a path P of order 3, and a 3-matching, joining every vertex of T to a separate triangle of $S_{m-4,3}$.

At this point we apply again the above argument, completing the proof of Theorem 2. $\square$

Concluding remarks

Theorem 3 in [10] gives a result more general than just inequality (1):

Theorem 7 *Let G be a graph of order n with $\mu(G) = \mu$. If G has no $K_{2,k+1}$ for some $k \geq 1$, then*

$$\mu^2 - \mu \leq t(n-1).$$

Equality holds if and only if every two vertices of G have exactly k common neighbors.

This theorem is sharper than Theorem 3 in [1], and for some values of n and k it is as good as one can get. However, in general, the maximal $\mu(G)$ of $K_{2,k+1}$-free graphs G of order n is not known at present.

Note that for $k > 1$, there may exist regular graphs with every two vertices having exactly k common neighbors: here is a small selection from [17]:

k	n	μ
2	16	6
3	45	12
4	96	20
5	175	30
6	36	15

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References

[1] L. Babai, B. Guiduli, Spectral extrema for graphs: the Zarankiewicz problem, preprint, (2007).

[2] B. Bollobás, *Modern Graph Theory,* Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998).

[3] B. Bollobás, V. Nikiforov, Cliques and the spectral radius, *J. Combin. Theory Ser. B.* **97** (2007), 859-865.

[4] P. Erdős, A. Rényi, V. T. Sós: On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

[5] C. Huneke, The Friendship Theorem, *Amer. Math. Monthly* **109** (2002), 192-194.

[6] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combin. Probab. Comp.* **11** (2002), 179-189.

[7] V. Nikiforov, The smallest eigenvalue of K_r-free graphs, *Discrete Math.* **306** (2006), 612-616.

[8] V. Nikiforov, A spectral condition for odd cycles, to appear in *Linear Algebra Appl.*

[9] V. Nikiforov, Eigenvalues and forbidden subgraphs I, *Linear Algebra Appl.* **422** (2007), 384-390.

[10] V. Nikiforov, Bounds on graph eigenvalues II, *Linear Algebra Appl.* **427** (2007) 183-189.

[11] V. Nikiforov, A spectral stability theorem for large forbidden graphs, submitted for publication. Preprint available at *http://arxiv.org/abs/0711.3485*

[12] V. Nikiforov, More spectral bounds on the clique and independence numbers, submitted for publication. Preprint available at *http://arxiv.org/abs/0706.0548*

[13] V. Nikiforov, A spectral Erdős-Stone-Bollobás theorem, submitted for publication. Preprint available at *http://arxiv.org/abs/0707.2259*

[14] V. Nikiforov, Spectral saturation: inverting the spectral Turán theorem, submitted for publication. Preprint available at *http://arxiv.org/abs/0707.2259*

[15] E. Nosal, Eigenvalues of Graphs, *Master's thesis,* University of Calgary, 1970.

[16] B. Papendieck, P. Recht, On maximal entries in the principal eigenvector of graphs, *Linear Algebra Appl.* **310** (2000) 129–138.

[17] G. Royle, Strogly regular graphs, *http://people.csse.uwa.edu.au/gordon/remote/srgs/*